Factoring Out the Impossibility of Logical Aggregation

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Abstract

According to a theorem recently proved in the theory of logical aggregation, any nonconstant social judgment function that satisfies independence of irrelevant alternatives (IIA) is dictatorial. This note shows that the strong and little plausible IIA condition can be replaced with a minimal independence assumption plus a Pareto-like condition. This new version of the impossibility theorem likens it to Arrow's and arguably enhances its paradoxical value.

1 Introduction

In political science and legal theory, the so-called *doctrinal paradox* refers to the observation that if a group of voters casts separate ballots on each proposition of a given agenda, and the majority rule is applied to each of these votes separately, the resulting set of propositions may be logically inconsistent. A mathematical theory of *logical judgment aggregation* has recently grown out of this straightforward point. Its method is to introduce a mapping from profiles of individual judgments to social judgments, where these judgments are formalized as sets of formulas in some logical language, and then investigate the effect of imposing conditions on this mapping. The

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major result obtained is an impossibility theorem that abstractly generalizes the doctrinal paradox (Pauly and van Hees, 2003; see also Dietrich, 2004). The theorem states that a mapping defined on a universal domain is dictatorial - i.e., collapses the social set into the set of a given individual whatever the profile - if and only if it is nonconstant and it satisfies independence of irrelevant alternatives (IIA). In the present context, the latter condition says that in order to decide whether a proposition belongs to the social set, it is enough to know whether that proposition belongs to the individual sets, regardless of how other propositions stand vis-à-vis these sets.

Given the Arrovian undertone of the chosen framework and conditions, this impossibility theorem seems puzzling. Arrow's social welfare function maps profiles of individual orderings to social orderings and is defined on a universal domain, and his IIA condition says that in order to decide what the social preference is between two alternatives, it is enough to know individual preferences over these alternatives, regardless of others. The two schemes of aggregation are intuitively related, and so are the two independence conditions. However, Arrow's conclusion that the social welfare condition is dictatorial (in the sense of reproducing one individual's strict preference) depends not only on IIA, but on a unanimity condition. The standard proof based on Arrow (1963) uses the weak Pareto condition. Wilson's (1972) important extension dispenses with this premiss, but needs an assumption in order to exclude antidictatorship, i.e., the rule which amounts to reversing one individual's strict preference. The theorem on logical judgment aggregation dispenses with any assumption of the kind. This should be surprising to any economist well acquainted with Arrow's work.

The present note offers a new variant of the theorem that will make it less mysterious. Still granting universal domain, it derives dictatorship from a IIA condition that is much weaker than the existing one, plus a unanimity condition without which the conclusion would not be reached. Both the earlier result and ours deliver essentially equivalent restatements of dictatorship; so what we do in effect is to factor out the strong IIA condition of the current theory. The proposed set of premisses appears to be preferable for two reasons. First, unanimity-preservation and independence-preservation are conceptually different properties for a social aggregate. If one can be hidden under the guise of the other, this is due to the peculiarities of the logical framework. One would better discount the influence of the framework by unpacking the compound condition in the set of premisses. Second, at the normative level, this IIA condition is hardly acceptable, whereas the suggested one deflates at least one of the telling objections against it. If the theory of logical judgment aggregation means to uncover a genuine paradox, not just a mathematical riddle, it must start from normatively defensible premisses. In the present, more Arrovian version, the paradox arises from one's combining two separate ideas that are attractive in isolation but happen to conflict within the chosen framework, i.e., independence (weakly stated) and unanimity-preservation (unrestricted).

2 The logical framework of aggregation

In the logical framework, a judgment consists in either accepting or rejecting a formula stated in some logical language. Like most previous writers, we will be concerned with the language of propositional logic. Accordingly, our set of formulas $\mathcal{L}$ is constructed from the propositional connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ ("not", "or", "and", "implies", "is equivalent to") and a set $\mathcal{P}$ of distinct propositional variables (p.v.) $p_1, ..., p_m, ...$ serving as building blocks. Theorem 1 is proved for $m \geq 3$, but the assumption is just for simplicity; more on this below. Literals, i.e., those formulas which are either p.v. or negations of p.v., will be denoted as follows: for any $p \in \mathcal{P}$, $\widetilde{p}$ means either p or $\neg p$, and $\neg\widetilde{p}$ means $\neg p$ in the first case and p in the second.

The system of propositional logic defines a inference relation holding between formulas; the further notions a theorem and a contradiction follow derivately. (For a full treatment, economists may consult Stigum, 1990.) Our choice of a logical language is less restrictive than it seems because it implicitly covers more expressive languages in which propositional logic can be embedded isomorphically. A translation of Theorem 1 for these extensions would be an essentially mechanical affair.

The *agenda* is the nonempty subset $\Phi \subseteq \mathcal{L}$ of formulas representing the actual propositions on which the n individuals and society pass judgment. We denote $\Phi \cap \mathcal{P} = \Phi_0$. Clearly, the slimmer the agenda, the more striking the impossibility theorem. Contrary to the original authors and Nehring and Puppe (2005), we leave out refinements in this direction. We will explain after the proof of Theorem 1 what it exactly needs to assume on Φ; for the time being, the reader is welcome to take $\Phi = \mathcal{L}$.

The theory does not investigate aggregative rules for judgments in isolation, but rather for judgment *sets*, where definite restrictions are imposed on what counts as such a set. Technically, it is any nonempty subset $B \subseteq \Phi$ that is consistent, as well as complete in the following relative sense: for any $\varphi \in \Phi$, either φ or $\neg\varphi$ belongs to B. This implies that B is deductively closed relative to Φ, i.e., for all $\varphi \in \Phi$ and $C \subset B$, if φ can be inferred from C by propositional logic, then $\varphi \in B$.

A *social judgment function* is any mapping

$$F : (A_1, ..., A_n) \mapsto A,$$

where the A_i, $i \in N = \{1, ..., n\}$, and A are judgment sets for the individuals and society, respectively. We denote $F(A_1, ..., A_n)$ by A, $F(A'_1, ..., A'_n)$ by A', etc. A social judgment function is *dictatorial* if there is $j \in N$ - the *dictator* - such that for all $(A_1, ..., A_n)$ in the domain,

$$A_j = F(A_1, ..., A_n).$$

Notice that Arrow's concept is effectively weaker since it does not amount to a projection. We say that j is a *dictator for* $(A_1, ..., A_n)$ if $F(A_1, ..., A_n) = A_j$, and that F is *dictatorial profile by profile* if there exists a dictator for each $(A_1, ..., A_n)$.We now introduce conditions on F. It will be a maintained assumption that F satisfies *Universal Domain*, i.e., $F : D^n \to D$, where D is the set of all possible judgment sets.

Axiom 1 *(Systematicity)*

$$\begin{aligned} \forall \varphi, \psi &\in \Phi, \forall (A_1, ..., A_n), (A'_1, ..., A'_n) \in D^n \\ [\varphi \in A_i \Leftrightarrow \psi \in A'_i, i = 1, ..., n] &\Rightarrow [\varphi \in A \Leftrightarrow \psi \in A'] \end{aligned}$$

Axiom 2 *(Independence of Irrelevant Alternatives)*

$$\begin{aligned} \forall \varphi &\in \Phi, \forall (A_1, ..., A_n), (A'_1, ..., A'_n) \in D^n \\ [\varphi \in A_i \Leftrightarrow \varphi \in A'_i, i = 1, ..., n] &\Rightarrow [\varphi \in A \Leftrightarrow \varphi \in A'] \end{aligned}$$

Axiom 3 *(Independence of Irrelevant Propositional Alternatives)*

$$\begin{aligned} \forall p \in \mathcal{P} \cap \Phi, &\forall (A_1, ..., A_n), (A'_1, ..., A'_n) \in D^n \\ &[p \in A_i \Leftrightarrow p \in A'_i, i = 1, ..., n] \Rightarrow [p \in A \Leftrightarrow p \in A'] \end{aligned}$$

The three conditions are listed from the logically strongest to the weakest. *Systematicity* requires that two formulas be treated alike by society if they draw the support of exactly the same people, even if these formulas refer to semantically unrelated items. Although this was the condition assumed in the first place (List and Pettit, 2002, show that it is incompatible with an anonymity requirement), it is quite obviously unattractive. Take a two-individual society in which 1 judges that the budget should be balanced, 2 disagrees, and the social judgment endorses 1. Then, if 1 also judges that marijuana should be legalized, and 2 disagrees again, the social judgment should endorse 1 again. Economists will recognize that this is a neutrality condition in the style of those of social choice theory and that it is no more appealing here than it is there (see Samuelson's 1977 witty criticism of the social-choice-theoretic condition).

Instead of permitting variations in both the profile $(A_1, ..., A_n)$ and formula φ, *Independence of Irrelevant Alternatives* fixes the formula and allows only the profile to vary; in this way, it avoids the confounding of semantic contents that spoils the earlier condition. It singles out the requirement contained in Systematicity that the social judgment on φ should depend only on the individual judgments on φ. Exactly as for Arrow's condition, the best normative defence for this restriction is that it prevents some possible manipulations (see Dietrich, 2004, and Dietrich and List, 2004). However, the condition is open to a charge of irrationality. One would expect society to pay attention not only to the individuals' judgments on φ, but also to their reasons for accepting or rejecting this formula, and these reasons may be represented by other formulas than φ in the individual sets. Before deciding that two profiles call for the same acceptance or rejection, society should in general take into account more information than is supposed in the condition.

The new condition of *Independence of Irrelevant Propositional Alternatives* (IIPA) amounts to reserving IIA to propositional variables alone. Consider again the doctrinal paradox: the trouble comes from the assumption that the majority rule applies to molecular formulas and propositional variables alike, i.e., that this independent and even neutral rule dictates on the whole of Φ. In contrast, when restricted to propositional variables, independence becomes more acceptable *because these formulas represent primary data*. One can object to IIA being applied to $p \vee q$, where p represents "The budget should be balanced" and q "Marijuana should be legalized" on the ground that there are two propositions involved, and that society should know how each individual feels about either proposition, and not only about their disjunction. No similar objection arises when IIA is applied to either p or q in isolation since the reasons for accepting or rejecting them are beyond the expressive possibilities of the language. Of course, as we pointed out earlier, propositional logic can be embedded in more powerful logics, which analyze the propositional variables instead of taking them as primary data. In this more refined framewok, the irrationality charge would carry through to the formulas replacing p or q, but it would be deflated when one reaches the stage of the new building blocks .

The last condition to be introduced is a straightforward analogue of the Pareto principle:

Axiom 4 *(Unanimity) For all $\varphi \in \Phi$, and all $(A_1, ..., A_n) \in D^n$,*

$$\varphi \in A_i, i = 1, ..., n \Rightarrow \varphi \in A.$$

3 The impossibility of logical judgment aggregation

Pauly and van Hees's (2003) show that if F satisfies Independence of Irrelevant Alternatives and is not a constant mapping, F is dictatorial. Dietrich's (2004) variant is specially adapted to a finite $\mathcal{P}$, and a simpler form of the impossibility theorem, which Pauly and van Hees also establish in generalizing List and Pettit (2002), states that if F satisfies Systematicity, F is dictatorial. The assumptions on Φ in all results are compatible with our provisional assumption of a maximal agenda $\Phi = \mathcal{L}$.

If one replaces the previous two conditions by Independence of Irrelevant Propositional Alternatives, the unpleasant conclusion vanishes. This can be seen by considering antidictatorship. We say that F is *antidictatorial* if there is j - the *antidictator* - such that for all $p \in \Phi_0$, and all $(A_1, ..., A_n) \in D^n$,

$$p \notin F(A_1, ..., A_n) \Leftrightarrow p \in A_j.$$

It is easy to check that this equivalence defines a social judgment set for each profile. (Observe that it would not if one required society to negate the molecular formulas of A_j as well as its propositional variables.) Clearly, IIPA is satisfied. To show that an antidictatorial F violates IIA, take

$$
\begin{aligned}
A_1 &= \{p_1, \neg p_2, ...\}, A_2 = \{\neg p_1, p_2, ...\}, \\
A_1' &= \{p_1, p_2, ...\}, A_2' = \{\neg p_1, p_2, ...\},
\end{aligned}
$$

and suppose that 1 is the antidictator. Then, $A = \{\neg p_1, p_2, ...\}$ and $A' = \{\neg p_1, \neg p_2, ...\}$, so that $p_1 \vee p_2 \in A$, $p_1 \vee p_2 \notin A'$. But $p_1 \vee p_2$ has the same pattern of acceptance in the two profiles, which contradicts IIA. By a similar argument, 2 cannot be the antidictator.

The example also shows that that IIPA does not imply Unanimity. When the two conditions are combined, dictatorship reappears.

Theorem 1 *If F satisfies IIPA and Unanimity, F is dictatorial.*

Proof. We first show that Unanimity implies that F is dictatorial profile by profile. Suppose by way of contradiction that there is $(A_1, ..., A_n) \in D^n$ such that $A \neq A_i$, for all $i = 1, ..., n$. By the maximality property of jugement sets, they differ from each other if and only if there is a propositional variable on which they differ. That is to say, for each i, there is $p_i \in \Phi_0$ such that

$$\widetilde{p}_i \in A_i \text{ and} \neg \widetilde{p}_i \in A.$$

By deductive closure, $\varphi = \vee_{i=1,...,n} \widetilde{p}_i \in A_i$, $i = 1, ..., n$, and Unanimity implies that $\varphi \in A$. By maximality, one of the literals $\widetilde{p}_i$ must be in A, which contradicts the assumption.

Now, we derive the following Positive Responsiveness property. Take any $p \in \Phi_0$, and any two profiles $(A_1, ..., A_n)$, $(A_1', ..., A_n') \in D^n$, such that (i) $\widetilde{p} \in A = F(A_1, ..., A_n)$, (ii) for at least one j, $\widetilde{p} \notin A_j$ and $\widetilde{p} \in A_j'$, (iii) for no i, $\widetilde{p} \in A_i$ and $\widetilde{p} \notin A_i'$. (The last two conditions say that $\widetilde{p}$ does not disappear from any individual judgment set and appears in at least one.) Then, $\widetilde{p} \in A' = F(A_1', ..., A_n')$.

To derive Positive Responsiveness, we consider any non-empty subset I of the set of individuals N and suppose that (i), (ii), (iii) hold:

$$\neg \widetilde{p} \in A_i,\ i \in I;\ \widetilde{p} \in A_i,\ i \in N \diagdown I;\ \neg \widetilde{p} \in A_i',\ i \in J \subset I;\ \widetilde{p} \in A_i',\ i \in N \diagdown J.$$

Suppose by way of contradiction that $\neg \widetilde{p} \in A'$. Since there is a dictator for each profile, it must be one of the $i \in N \diagdown I$ for $(A_1, ..., A_n)$, and one of the $i \in J$ for $(A_1', ..., A_n')$. Now, take any propositional variable $q \neq \widetilde{p}$, $\neg \widetilde{p}$; by

IIPA, we can make any assumption we wish on the acceptance or rejection of q in the two profiles. Let us suppose that

$$q \in A_i \cap A_i',\ i \in I;\ \neg q \in A_i \cap A_i',\ i \in N \diagdown I.$$

Because exactly the same i accept q in $(A_1, ..., A_n)$ and $(A_1', ..., A_n')$, IIPA implies that

$$q \in A \Leftrightarrow q \in A',$$

a contradiction because the two profile dictators cannot agree on q (since one belongs to $N \diagdown I$ and the other to $J \subset I$).

We now set out to prove that IIPA and Unanimity together imply the following Limited Systematicity property: for all $(A_1, ..., A_n), (A_1^{'}, ..., A_n^{'}) \in D^n$, and all $p, q \in \Phi_0$,

$$\begin{aligned} (*)\ \ [p \in A_i \Leftrightarrow q \in A_i', i = 1, ..., n] &\ \ \Rightarrow\ \ [p \in A \Leftrightarrow q \in A'] \text{ and} \\ (**)\ \ [p \in A_i \Leftrightarrow \neg q \in A_i', i = 1, ..., n] &\ \ \Rightarrow\ \ [p \in A \Leftrightarrow \neg q \in A'] . \end{aligned}$$

Take $p \neq q$ and suppose that the assumption of (*) holds for $(A_1, ..., A_n), (A_1^{'}, ..., A_n^{'})$. There exists $(B_1, ..., B_n)$ s.t.

$$p \in A_i \Leftrightarrow p \in B_i, \text{ and } q \in A_i' \Leftrightarrow q \in B_i, i = 1, ..., n.$$

Assume that $p \in A$. From IIPA, $q \in B = F(B_1, ..., B_n)$. Because there is a dictator for $(B_1, ..., B_n)$, we conclude from the above equivalences that $q \in B$, and applying IIPA again, that $q \in A'$. The proof of (**) is similar. Now, suppose that $p = q$. Statement (*) is simply IIPA. In order to derive (**), we assume that

$$p \in A_i \Leftrightarrow \neg p \in A_i',\ i = 1, ..., n,$$

and apply what has just been proved for distinct variables. There exist $r \in \Phi_0$, $r \neq p$, and a profile $(B_1, ..., B_n)$ such that

$$p \in A_i \Leftrightarrow \neg r \in B_i,\ i = 1, ..., n.$$

If $p \in A$, then $\neg r \in B$, hence $\neg p \in A'$, as desired.

Now, fix any $p \in \Phi_0$. We can find in D^n a sequence of profiles $\left(A_j^i\right)_{j=1,...,n}$, $i = 1, ..., n$, with the following properties:

$$\begin{array}{llll}
\neg p\in A_1^1 & p\in A_2^1 & \ldots & p\in A_n^1\\
\neg p\in A_1^2 & \neg p\in A_2^2 & p\in A_3^2\ldots & p\in A_n^2\\
\ldots & \ldots & \ldots & \ldots\\
\neg p\in A_1^n & \neg p\in A_2^n & \neg p\in A_3^n\ldots & \neg p\in A_n^n
\end{array}.$$

Denote by $(A^i)_{i=1,\ldots,n}$ the associated sequence of social judgment sets. Define i^* to be the first i such that both

$$\neg p\in A_{i^*}^{i^*} \text{ and } \neg p\in A^{i^*}.$$

Unanimity ensures that i^* exists. We will prove that i^* is a dictator, and for this, we need another preparatory step.

We aim at showing that there exists a profile $(B_1,...,B_n)$ such that $\neg p\in B$ and

$$(*)\ \neg p\in B_{i^*},\ p\in B_i,\ i\neq i^*.$$

For $i^*=1$, it is enough to take the first line in $\left(A_j^i\right)_{j=1,\ldots,n}$. For $i^*\geq 2$, we need to consider three sets of individuals:

$$I=\{1,...,i^*-1\},\ J=\{i^*\},\ K=\{i^*+1\},$$

the first two of which are never empty. Assume by way of contradiction that $p\in B$ for all profiles satisfying $(*)$. In particular, $p\in B'$ for $(B_1',...,B_n')$ satisfying $(*)$ and

$$\begin{array}{rcl}
\neg q & \in & B_j', j\in I\cup J; q\in B_j', j\in K,\\
r & \in & B_j', j\in I; \neg r\in B_j', j\in J\cup K,
\end{array}$$

where we choose two distinct propositional variables $q,r\neq p$. Also, $p\in B''$ for $(B_1'',...,B_n'')$ satisfying $(*)$ and

$$\begin{array}{rcl}
\neg q & \in & B_j'', j\in I; q\in B_j'', j\in J\cup K\\
r & \in & B_j'', j\in I; \neg r\in B_j'', j\in J\cup K.
\end{array}$$

From Limited Systematicity, we conclude that $\neg q\in B'$ and $q\in B''$. (The first conclusion uses the i^*-th line in $\left(A_j^i\right)_{j=1,\ldots,n}$ and replaces p by q; the second conclusion uses the (i^*-1)-th line, again replacing p by q.) Now, the first conclusion means that the dictator for $(B_1',...,B_n')$ is some $j\leq i^*$, and the second, that the dictator for $(B_1'', ..., B_n'')$ is some $j \geq i^*$. In either case, $j = i^*$ is excluded since $p \in B' \cap B''$ and $\neg p \in B_{i^*}' \cap B_{i^*}''$. So we are left with two statements for r - i.e., $r \in B'$ and $r \notin B''$ - which conflict with IIPA.

To sum up, we have fixed p and shown that there exist an individual i^* and a profile such that i^* alone accepts $\neg p$ and the social set endorses i^* on this particular formula. By Limited Systematicity, the widely more general conclusion holds that for *all* propositional variables q and *all* profiles, if i^* alone accepts $\widetilde{q}$, then the social set contains $\widetilde{q}$. It remains to consider those profiles which, for some q, have more individuals than just i^* accepting $\widetilde{q}$. To handle such a $(C_1, ..., C_n)$, we compare it with a profile $(C_1', ..., C_n')$ in which i^*alone accepts $\widetilde{q}$. Positive Responsiveness ensures that $\widetilde{q} \in C$. Having thus shown that for all $q \in \Phi_0$, and all $(A_1, ..., A_n) \in D^n$,

$$q \in A \Leftrightarrow q \in A_{i^*}.$$

we conclude from the maximality of individual sets and the deductive closure of social sets that the same holds for all $\varphi \in \Phi$. This completes the proof that F is dictatorial, with i^* as a dictator. ■

The reader will have observed the social-choice-theoretic undertones of the proof. In the classic terminology, the last but one paragraph states a "semi-decisiveness" property for a particular formula and a particular profile, and the last paragraph performs the remaining steps of proving "decisiveness" for every formula and every profile. Notice the rôle of the assumption that individual and social sets are not only consistent and deductively closed, but also *maximal.* Gärdenfors (2005) has stressed that this is a strong assumption to make, and it turns out to be crucial at several stages of the proof.

It is now possible to describe our class of agendas Φ more carefully. Only the following assumptions are needed: (i) Φ is closed under propositional variables, i.e., if $\varphi \in \Phi$ and $p \in \mathcal{P}$ occurs within φ, then $p \in \Phi$; (ii) if $p \in \Phi_0$, then $\neg p \in \Phi$; (iii) Φ is closed under arbitrary n-disjunctions of literals, i.e.,

$$\widetilde{p}^{(1)}, ..., \widetilde{p}^{(n)} \in \Phi \Rightarrow \vee_{i=1,...,n} \widetilde{p}^{(i)} \in \Phi.$$

The last assumption comes into play in the first step of the proof. Inspection shows that it can be replaced by (iii'): Φ is closed under arbitrary n-conjunctions of literals. Pauly and van Hees's (2003) class of agendas also require (i) and (ii), but their third clause relates to 2-conjunctions of literals, hence coincides with (iii') only when there are two individuals. Dietrich's

(2004) assumption for the finite case involves the atoms of $\mathcal{L}$ and is not easily comparable with Pauly and van Hees's and ours.

The assumption that $\mid \mathcal{P} \mid \geq 3$ can be relaxed. Using an inductive argument that would make the middle of the proof more complex, we can derive the theorem for propositional languages with two distinct propositional variables.

In the case $n = 2$, there is quick termination of the proof after the stage of proving Limited Systematicity.

Proof. Having shown that F is dictatorial profile by profile and satisfies Limited Systematicity (1st and 3d paragraphs), we conclude by the following reductio. If neither 1 nor 2 were a dictator, there would exist (A_1, A_2), $(A'_1, A'_2) \in D^2$ such that

$$A_1 \neq A_2,\ A'_1 \neq A'_2,\ A = A_1,\ A' = A'_2.$$

So there would exist $p, q \in \Phi_0$ such that $\widetilde{p} \in A_1, \widetilde{p} \notin A_2$, and $\widetilde{q} \in A'_1, \widetilde{q} \notin A'_2$, whence $\widetilde{p} \in A, \widetilde{q} \notin A'$. However, the last two statements contradict Limited Systematicity when it is applied to the first four.■

We close the analysis with an example of a well-behaved social judgment function F^*. For any $(A_1, ..., A_n) \in D^n$, we define $A^* = F^*(A_1, ..., A_n)$ in two steps. First, we define $B^* \subset A^*$ by two properties, i.e., (i):

$$\forall p \in \Phi_0,\ p \in B^* \Leftrightarrow N_p \geq N_{\neg p},$$

where N_p, $N_{\neg p}$ denote the numbers of $i \in N$ such that $p \in A_i$ and $\neg p \in A_i$, respectively, and (ii):

$$p \notin B^* \Leftrightarrow \neg p \in B^*.$$

Second, we define A^* to be the deductive closure of B^*. This construction delivers judgment sets for all profiles. It amounts to applying the majority rule to propositional variables alone - with a tie-breaking clause in the case where there are as many supporters of p as there are of $\neg p$ - and then let society draw the logical consequences. This is a special case of the so-called *premiss-based procedure* where the set of premisses is Φ_0 (more on the general procedure in Dietrich, 2004).

When restricted to Φ_0, the social judgment function F^* satisfies the familiar properties of the majority rule, which are translated here as Systematicity, Unanimity, and an Anonymity condition that we do not spell out formally. Trouble occurs only when these properties are extended to Φ. Indeed, for $n=3$, $\mathcal{P}=\{p_1,p_2,p_3\}$, and $\Phi=\mathcal{L}$, take

$$A_1=\{p_1,\neg p_2,p_3,...\}\,,\ A_2=\{p_1,p_2,\neg p_3,...\}\,,\ A_3=\{\neg p_1,p_2,p_3,...\}\,,$$

so that $A^*=\{p_1,p_2,p_3,...\}$, and suppose now that we add to the definition

[redacted]

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